OPEN-ENDED QUESTIONS AND THE PROCESS STANDARDS

Educating students—for life, not for tests—implies incorporating open-ended questions in your teaching to develop higher-order thinking.

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All societies need citizens who can solve complex problems and apply knowledge in a variety of contexts as well as citizens who can work collaboratively to solve problems and communicate solutions to mathematics education stakeholders. We must educate students to use NCTM's Process Standards (NCTM 2000) and move beyond being able to work routine exercises on standardized tests. We are not educating students for tests; we are educating them for life. All stakeholders need to see this broader picture and support teachers in this broader purpose.

As a high school mathematics teacher and mathematics teacher educator, I have used open-ended questions as part of my own teaching practice. Open-ended questions, as discussed here, are questions that can be solved or explained in a variety of ways, that focus on conceptual aspects of mathematics, and that have the potential to expose students' understanding and misconceptions. When working with teachers who are using open-ended questions with their students for the first time, I have found that they learn a considerable amount, as I did, about what their students both know and do not know—much more than what they knew before they started using open-ended questions. Teachers are almost always surprised, a little disappointed, but often excited about what they discover.

I will share some student responses from the class of a high school mathematics teacher with whom I have worked. Ms. Yoder has high expectations of her students. Her students work together to solve problems that require a high level of cognitive demand; the kind of thinking necessary to solve the problems forces students to build "connections to underlying concepts and meaning" (Stein et al. 2009, pp. 1–2). After having her students work some of the problems presented here, Ms. Yoder commented, "I was dismayed at the lack of depth and the simplicity of some students' responses. I have always felt that I teach on a conceptual level, and I do a lot of listening to students' conversations to assure myself that the level of understanding meets my hopes and expectations. . . . But I have rarely required my students to write about mathematics." After using these problems with her students, Ms. Yoder reflected, "Asking these questions made me rethink my means of assessing students."

When we think about assessment in this era of No Child Left Behind, we often think about high-stakes standardized tests, which are typically multiple-choice tests. So much of what happens in mathematics classes is focused on preparing students to succeed on these tests. As I work with teachers, they express high levels of anxiety about making sure that their students are prepared for these high-stakes tests. Mathematics education stakeholders—including teacher educators, administrators, teachers, students, and parents—need to reflect on what standardized tests can and cannot measure. Even more important, they must evaluate the educational significance of those ideas that standardized tests cannot assess.

NCTM's Process Standards—Problem Solving, Reasoning and Proof, Communication, Connections, and Representation—are difficult to assess with multiple-choice tests. For example, one aspect of the Communication Standard requires students to "communicate their mathematical thinking coherently and clearly to peers, teachers, and others" (NCTM 2000, p. 60). This standard cannot be assessed through multiple-choice questions.

If we do not teach what is not tested, what are the implications of not preparing students to meet these Process Standards? Consider the following statement by a BC Calculus student:

> My experience in the past—and not to hate on the teachers I've had—but they've never really encouraged us to think. It's all been cookie-cutter questions, even with word problems. I remember my algebra 1 teacher—she had a little trick for everything. Of course, I don't remember the trick now, and I don't remember why I was doing it. I felt like there were a lot of shortcuts, and I was never really taught why we were using them. So I memorized everything, which is what I've been doing ever since (Stockton 2010).

This student was lamenting her inability to solve a complex problem. A student capable of handling the difficult BC Calculus curriculum expressed her own disappointment that the focus of her education had been procedural.

As teachers struggle to ensure that students are able to answer questions correctly on procedural tests, many are desperate to find ways to help them remember strategies and steps to find correct solutions. However, problems that people encounter in everyday life and careers rarely require rote application of procedures.

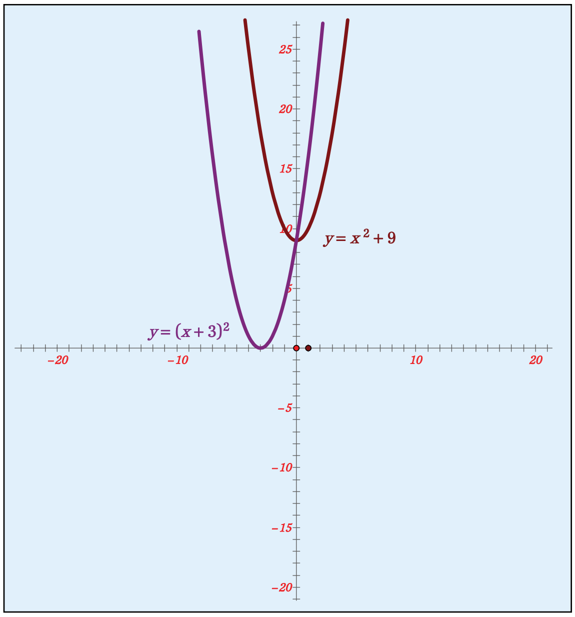

Fig. 1 Students might also argue that $y = (x + 3)^2$ and $y = x^2 + 9$ are, respectively, horizontal and vertical shifts of $y = x^2$.

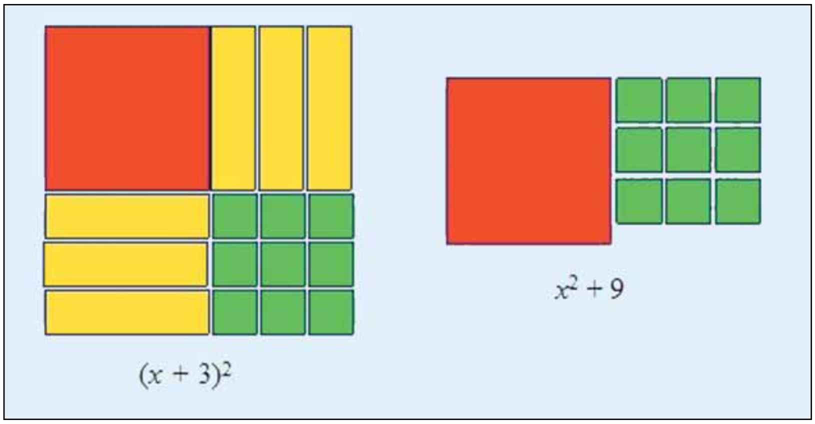

Fig. 2 Algebra tiles geometrically represent the statement $(x + 3)^2 \neq x^2 + 9$.

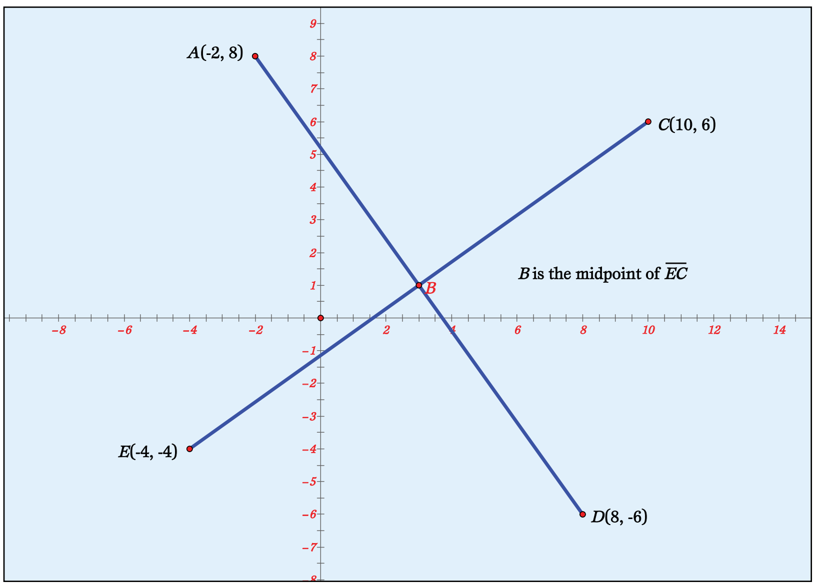

Fig. 3 Slopes, the Pythagorean theorem, congruent triangles, and dot products may all be used to show that $\angle ABC$ is a right angle.

OPEN-ENDED QUESTIONS CAN FOCUS INSTRUCTION ON PROCESS STANDARDS

Using NCTM's Process Standards as a guide, teachers can make questions more open and more focused on conceptual understanding.

Consider this traditional question:

> Expand $(x + 3)^2$.

We could revise this question in several ways. If we wanted to address the Communication Standard, we could ask students to explain how they determined their answer. We could take the question even further to incorporate other Process Standards. We could capitalize on a common student error and ask students to explain why $(x + 3)^2 \neq x^2 + 9$. Now we have expanded the question to include the Communication Standard and the Reasoning and Proof Standard. We could go even further to address the Representation Standard by asking students to give two or three different explanations of why $(x + 3)^2 \neq x^2 + 9$.

A typical first explanation that students provide is this:

> $(x + 3)^2$ means $(x + 3)(x + 3)$. I can use the distributive property to multiply these two binomials so I get $x^2 + 3x + 3x + 9$, which equals $x^2 + 6x + 9$, which is not the same as $x^2 + 9$.

Asking students for another explanation forces them to consider a different representation. For example, they might choose a numerical representation and substitute a numerical value for x. Their explanation might then be something like this:

> Let $x = 2$. $(x + 3)^2 = (2 + 3)^2 = 25$.
> $x^2 + 9 = 2^2 + 9 = 13$. Because $25 \neq 13$, $(x + 3)^2 \neq x^2 + 9$.

Students could also consider a graphical representation and show that the graphs of $y = (x + 3)^2$ and $y = x^2 + 9$ are different (see **fig. 1**). They could even consider the problem geometrically by using algebra tiles (see **fig. 2**).

If we teachers intentionally consider NCTM's Process Standards when writing questions, we can make sure that students are required to use the processes. With this particular question, we also counter a common student error in several ways. By seeing multiple representations, students are more likely to avoid the error later on.

What Process Standards might students use to solve the following problem?

> Use three different methods to show that $\angle ABC$ is a right angle. Explain your reasoning. (See **fig. 3** for solution.)

In solving this problem, students might use the midpoint formula to determine the coordinates of point B and then show that $AB^2 + BC^2 = AC^2$. In this way, they verify that triangle ABC is a right triangle because its sides satisfy the Pythagorean theorem and that, therefore, angle ABC is a right angle. Or, using the distance formula, students might show that $AE = AC$; then, using the side-side-side postulate, they can show that $\triangle ABC \cong \triangle ABE$. Therefore, $\angle ABE \cong \angle ABC$ because corresponding parts of congruent triangles are congruent. Because these two angles are congruent and form a linear pair, they must be right angles.

Still another way to solve this problem is to compute the slopes of $\overleftrightarrow{EC}$ and $\overleftrightarrow{AD}$ and show that their product is –1. More advanced students can demonstrate the dot product of [7, 5] (the rectangular vector from B to C) and [–5, 7] (the rectangular vector from B to A) is 0, making the two vectors orthogonal (perpendicular).

When students are required to provide multiple solutions, they often use a variety of representations. As they explain their reasoning, they are communicating. Although students need to rely on some procedural knowledge to answer this problem, they have to decide which procedures would apply to it. They are not provided with a step-by-step procedure; consequently, they are involved in problem solving as well as reasoning and proof. They are making connections among a variety of mathematical topics—slope, congruent triangles, midpoints, the distance formula, the Pythagorean theorem, and vectors.

WRITING OPEN-ENDED QUESTIONS

Open-ended questions can be written using various templates, several of which are discussed here. Teachers who are just beginning to use open-ended assessment can use these templates for creating their own questions. We provide examples of several types, and for one question of each type, we provide sample student responses.

Template 1: What's Wrong with This?

The earlier question about expanding $(x + 3)^2$ is an example of this type of question used to identify errors and misconceptions. We can ask students to identify errors and explain why they are errors. This template is useful for getting students to think critically about common misconceptions.

Some possible questions using this template follow:

1. Provide two different explanations as to why you cannot simplify the expression $(x + 3)/3$.
2. Bert was trying to graph $y = (x - 3)^2$. He said that he could simply shift the graph of $y = x^2$ three units to the left. Convince Bert that his method is incorrect.

(a)

Response A	In the graph, $X \leq 3$
Response B	sherri's solution is incorrect because she misinterpreted the inequality. The correct answer is $X \geq 3$ and $X \leq 5$ because a colored dot on the segment means equal to
Response C	you know that sherri's solution is incorrect by reading the graph. The arrow coming from 3, is pointing in the lesser direction. therefore the correct solution is $5 \leq X \leq 3$

(b)

Fig. 4 Sherri's solution (**a**) is incorrect. Typical student responses are shown in (**b**).

Fig. 5 Students are asked to provide a possible equation to match this graph.

3. Sherri claims that the solution set of the compound inequality $x \geq 3$ or $x \geq 5$ is shown in **figure 4**. Explain why Alaine's solution is incorrect. Provide the correct solution and explain how you know your solution is correct.

Question 3 was designed to counter the common student error of thinking that *or* always means that the arrows on the graph of a linear inequality should point in opposite directions. Of course, the correct solution set of the linear inequality is $x \geq 3$ because the *or* means one *or* the other *or* both. Therefore, any real number greater than or equal to 3 would be in the solution set.

None of the students who answered the question (even those whose solutions are not shown in **fig. 4b**) provided the correct solution. They focused on the direction of the inequality sign rather than on the meaning of the conjunction *or*. Student B appears to have some misconception about

Response A	Create a system of linear equations that has the solution (–2, 3). Explain how you determined your system. $y = (x+2)^2 + 3$ horizontal shift, +vertical shift
Response B	$y = -2x - 1 \rightarrow y = -2(-2) - 1$, $y = 4 - 1 \rightarrow y = 3 \rightarrow (-2, 3)$
Response C	$y = x + 5$ $y = -2x - 1$; $x + 5 = -2x - 1$; $3x = -6$; $x = -2$. I developed a system of linear equations with the solution (-2,3) then I get them equal to each other and solved for y.

Fig. 6 Students find it difficult to create a linear system when given the solution.

changing the direction of an inequality sign, an "equal to," and a "colored dot."

Template 2: Create an Example or a Situation

This form of question is similar to the form of the questions for the game show *Jeopardy*™. We give students some parameters and ask them to come up with an example or situation that fits the parameters. We give them the answer and have them come up with the question.

Some possible questions using this template follow:

1. Give a possible equation for the graph shown in **figure 5**. Explain how you determined your answer.
2. On a coordinate grid, plot and give the coordinates of four points that are the vertices of a rhombus. Explain how you know that your figure is a rhombus.
3. Create a list of ten different numbers whose median is 9. Explain how you know that the median is 9.
4. Give two complex numbers whose sum is $7 + 9i$. Explain how you know that your two numbers have the given sum.
5. Create a system of linear equations that has the solution (–2, 3). Explain how you determined your system.

The first time I used open-ended questions in my teaching, I included question 5 on an exam. Many students got every question correct except this one. The first section of the exam asked students to "solve these systems of linear equations by graphing"; the second section, to solve by substitution; the third section, to solve by elimination; and the fourth section, to solve by any method. Then I added this single open-ended question, and my students were thrown. I knew then that not only was I asking the wrong questions; I was also focusing my instruction on the wrong things. My students could follow procedures that I taught them, but they did not really know what a system of linear equations was or what a solution of a system of linear equations was.

Ms. Yoder's students' responses are informative (see **fig. 6**). Student A describes shifts of graphs of quadratic functions, whereas student B found a single line that contained the point (–2, 3). I think that students A and B would do just fine on a standardized test about systems of linear equations. Like my students who got every problem correct on my test except this one, these students might be able to answer standard questions without really understanding what a system of linear equations is. After reading these responses, however, I am much more confident that student C has a deeper understanding of systems of linear equations than either of the other two students.

Template 3: Who Is Correct and Why?

This form of open-ended question—Who is correct and why?—can be used to set up two opposing arguments. Then students can defend one or the other argument.

Some possible questions using this template follow:

1. Lucinda thinks that the grades in mathematics class should be calculated using the mean. Norm thinks that the grades should be calculated using the median. With whom do you agree and why?
2. Daniella is thinking about a particular quadratic function. Terry says that if Daniella told him the zeros of the function, he could tell her the equation of the function. Daniella maintains that Terry would need more information. Who is correct and why?
3. Candace said that if she solves the same system of linear equations as Jermaine, they could get two different answers and both be correct. Jermaine disagreed, saying that if they got two different answers, one of them must be incorrect. Who is correct and why?

Response A	Candice is correct because depending on the function if you have to take a square root there are two possible correct answers.
Response B	Candice is correct because the two answers are part of the graph therefore they are correct
Response C	Jermaine, because the equations are linear, meaning only one intersect point

Fig. 7 Only one of these students fully understands the question.

Question 3 also was designed to get at the meaning of the solution of a system of linear equations. From the responses, it appears that only student C (see **fig. 7**) seems to understand the main point of the question—that two lines can intersect only in one point.

A Caution about These Templates

The templates presented here can be useful in giving teachers a place to start when writing open-ended questions, but teachers must be cautious when using them. Just because a question fits a template does not necessarily mean that the question is open ended or of high quality.

For example, we could ask the earlier question in this way:

> Jasmine solved $x + 3 = 5$ and got $x = 2$. Stuart solved $x + 3 = 5$ and got $x = 8$. Who is correct and why?

This form of the question is no different from asking the traditional question "Solve $x + 3 = 5$ for x." The formulation does not involve the conceptual underpinnings of equation solving.

PREPARATION FOR LIFE

Teachers are under more pressure than ever to ensure that students perform well on standardized tests. Consequently, many are using more multiple-choice questions to prepare their students. School districts are using benchmark testing to assess students' progress toward meeting standards and prepare them for accountability tests. These are all perfectly reasonable strategies, but mathematics education stakeholders must keep in mind the limits of these accountability tests. If we think about the purpose of schooling from a broader perspective and about preparing students to solve the kinds of problems that they will encounter in society—not just about preparing them for standardized tests—we need different strategies.

Open-ended questions can help teachers focus their instruction and assessment on NCTM's Process Standards and on reasoning and sense making, which really is the heart of mathematics. Moreover, responses to open-ended questions give teachers so much more information about students' ways of thinking and misconceptions, and these can provide important avenues for further investigation of mathematics. When students answer higher-order questions driven by the Process Standards and focused on meaning, they will be prepared for any test we give them—in school or in life.

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